Explicit Construction of Inverse Functions of Polynomials

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Abstract

This paper shows how to build inverse functions of polynomials, from a theoretical point of view. In other words, given a polynomial function $y = P(x) = a_0 + a_1x + \cdots + a_mx^m$, with $a_i \in \mathcal{R}, 0 \leq i \leq m$, and a real number u so that $P'(u) \neq 0$, we get an explicit function $F_P(y)$ that satisfies $x = F_P(P(x))$ around $x = u$. The procedure that we have followed to achieve $F_P(y)$ is to compute its Taylor series and its domain of convergence. If $u = 0$, $F_P(y)$ will be denoted by $f_P(y)$.

For practical purposes, we are exploring the possibilities of the functions $F_P(y)$ in order to solve polynomial equations and algebraic systems.

Keywords: Polynomial functions, quadratic formula, Cardan-Tartaglia Formula, Galois Theorem, Adomian Method, Decomposition Method, inverse of polynomials.

1 Introduction

The search for solutions for a polynomial equation:

$$0 = a_0 + a_1x + \cdots + a_mx^m \tag{1}$$

has contributed to the development of mathematics throughout centuries, ([1], [2], [3], [4]), since Sumerian (third millennium B.C.), Babylonian (second millennium B.C.) and Egyptian (second millennium B.C.) times until nowadays.

Particularly, the study of quadratic, cubic and quartic equations, ([5], [6], [7], [8], [9]), has motivated the introduction of some important concepts of mathematics such as irrational and complex numbers. The Galois Theory was motivated by this same problem of solving (1). This theory included the proof of the non-existence of solutions of (1) by radicals, but also introduced the ideas of groups and ideals, that motivated the development of modern algebra. Finally, (1) has influenced the earlier development of numerical computing.

Polynomial equation (1) arises in many fields of the sciences, engineering, economy and statistics.

In such areas available algorithms for solving (1) ([10], [11], [12], [13], [14], [15], [16], [17], [18]) are effective for the "average"polynomial of a small or moderate degree, but are heuristic in general. They do not converge to the roots of (1) unless some suitable initial values are given, and no global procedures are provided in order to find such a convenient approximation, for a general polynomial equation.

When the degree, m, grows beyond 10 or 20, the present-day necessities for solving (1) become more and more extended, but above all in the important case of *Computer Algebra*, where solving (1) for larger m is needed. Here is where equation (1) keeps playing its role nowadays, both as a research problem and as a part of computing tasks.

As it was expected, all these problems are increased in the case of polynomial systems.

As a potencial application, we are exploring the possibilities of inverse functions of polynomials, $F_P(y)$, introduced in this paper, in order to solve the problems of convergence and initial values, that we have outlined above, for both cases: polynomial equations and polynomial systems. Some examples of such applications may be found in ([19]).

Finally, we campare with the Adomian decomposition method [20]. If we consider the polynomial equation $x^m + px + q = 0$, and its inverse function $f_P(y)$, provided in this paper, then the n first terms of the series $f_P(0)$, coincide with the Adomian polynomials of degree $n-1$ for solving polynomial equations.

This paper is organized as follows. From the analytic solution, H, of functional equation (2), section 2 deals with the construction of the inverse function of $P(x)$ around $x=0$: $f_P(y)$. In section 3, we compute the inverse of $P(x)$ around any $x=u$, with $P'(u)\neq 0$, by applying the results of section 2 to the polynomial $P_u(z) = P(z+u)$, where we have substituted the new variable $z = x-u$ by the original variable x. Finally, section 4 illustrates how quadratic and Cardan-Tartaglia formulas are particular cases of functions $F_P(y)$, that we introduce in this paper.

Throughout this paper all the sets of subscripts or superscripts will be considered as non-negative integer numbers; the degree of $P(x)$ will be denoted by m; and $H^{i_1,\ldots,i_{m-1}}_{X_1,\ldots,X_{m-1}}(\overline{0})$ (the partial derivative of H with respect to X_1, i_1 times, $\ldots$, and with respect to X_{m-1}, i_{m-1} times, valued at zero), by $H^{\overline{i}}_{\overline{X}}(\overline{0})$.

2 Inverse function around $y = a_0$

We begin by introducing functional equation (2), whose solution is the central point of this section.

Definition 1 *Let $m \geq 2$ be an integer number. Let $x = (x_1, \cdots, x_{m-1})$ be a vector of $\mathcal{R}^{m-1}$ and $h : \mathcal{R}^{m-1} \to \mathcal{C}$ a function with $h(0,\cdots,0) = 1$. We define the functional equation:*

$$h(x) = 1 + x_1 h^2(x) + x_2 h^3(x) + \cdots + x_{m-1} h^m(x) \tag{2}$$

Lemma 1 *Equation (2) is solvable.*

Proof. It is easy to see that the order relation $\ll$ in $\mathcal{C}$ defined by $r_i \ll r_j \Leftrightarrow$

$$\begin{cases} Real(r_i) + Imag(r_i) < Real(r_j) + Imag(r_j) & \text{if } Real(r_i) + Imag(r_i) \neq Real(r_j) + Imag(r_j) \\ Real(r_i) \leq Real(r_j) & \text{if } Real(r_i) + Imag(r_i) = Real(r_j) + Imag(r_j) \end{cases}$$

is a total order relation.

Given $x^0 = (x_1^0, \ldots, x_{m-1}^0) \in \mathcal{R}^{m-1}$, with $x^0 \neq (0, \ldots, 0)$, (2) defines the polynomial equation of degree p, with $2 \leq p \leq m$: $h(x^0) = 1 + x_1^0 h^2(x^0) + x_2^0 h^3(x^0) + \cdots + x_{m-1}^0 h^m(x^0)$. Solving for $h(x_0)$, there are p solutions in $\mathcal{C}$, that we order in the following manner: $r_1^{(x_0)} \ll r_2^{(x_0)} \ll \cdots \ll r_p^{(x_0)}$. Then the p functions:

$$h_i : \mathcal{R}^{m-1} \to \mathcal{C}, \qquad h_i(x^0) = \begin{cases} r_i^{(x^0)}, & \text{if } x^0 \neq (0, \cdots, 0) \\ 1, & \text{if } x^0 = (0, \cdots, 0) \end{cases}$$

with $i = 1, \cdots, p$, are solutions of (2).

Lemma 2 provides a inverse function of (3), (4), from a solution, H, of (2).

Lemma 2 *Given the polynomial function:*

$$y = P(x) = a_0 + a_1 x + \cdots + a_m x^m \tag{3}$$

where $a_0, a_1, \cdots, a_m$ *are real numbers with* $a_1, a_m \neq 0$*. and let* H *be a solution of (2), then the function:*

$$f_P(y) = \frac{a_0 - y}{-a_1} H(X_1(y), \cdots, X_{m-1}(y)) \tag{4}$$

verifies:

$$x = f_P(P(x)) \tag{5}$$

where the functions $X_i(y)$*,* $1 \leq i \leq m-1$*, are:*

$$X_i(y) = \frac{(a_0 - y)^i a_{i+1}}{(-a_1)^{i+1}} \tag{6}$$

Proof. In fact, as H is a solution of (2), it follows that:

$$\begin{aligned} f_P(y) &= \frac{a_0 - y}{-a_1}\Big(1 + X_1(y) H^2(X_1(y), \ldots, X_{m-1}(y)) + \ldots \\ &+ X_{m-1}(y) H^m(X_1(y), \cdots + X_{m-1}(y))\Big) \end{aligned} \tag{7}$$

Substituting functions (6) in (7) one gets:

$$y = a_0 + a_1 f_P(y) + a_2 f_P^2(y) + \cdots + a_m f_P^m(y)$$

Therefore $x = f_P(y) = f_P(P(x))$ and (5) holds.

Here are the main steps in order to obtain $f_P(y)$ in an analytic way:

1. Theorem 1, based on Lemmas 3 and 4, computes the coefficients of Taylor series of H (16).
2. Inequality (17), in Lemma 5, provides an upper bound for coefficients (16).
3. Using (17), Theorem 2 proves that H is analytic in domain (20).
4. Finally, Theorem 3 gives $f_P(y)$ in an explicit way, by defining its series.

Lemma 3 *Let us consider polynomial function (3), function (4) and a positive integer number, n_0. Then the $(n_0+1)-th$ derivative of $f_P(y)$ with respect to y, valued at $y = a_0$, is given by the expression:*

$$\left.\frac{d^{(n_0+1)} f_P(y)}{dy^{n_0+1}}\right|_{y=a_0} \tag{8}$$

$$= \sum_{j=1}^{n_0} \sum_{\substack{i_1+\cdots+i_{m-1}=j \\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}} (-1)^j \frac{(2i_1+\cdots+mi_{m-1})!}{i_1!\cdots i_{m-1}!} \frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{1+2i_1+\cdots+mi_{m-1}}}$$

where $y = P(x)$.

Proof. Differentiating $x = f_P(P(x))$ with respect to x, we obtain:

$$\frac{df_P(y)}{dy} = \frac{1}{P'(x)}$$

Differentiating again n_0 times with respect to x, and solving for $d^{(n_0+1)} f_P(y)/dy^{n_0+1}$ it becomes:

$$\frac{d^{(n_0+1)} f_P(y)}{dy^{n_0+1}}$$

$$= \sum_{j=1}^{n_0} \left(\sum_{\substack{i_1+\cdots+i_{m-1}=j \\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}} (-1)^j \frac{(2i_1+\cdots+mi_{m-1})!}{(2!)^{i_1} i_1! \cdots (m!)^{i_{m-1}} i_{m-1}!} \right. \tag{9}$$

$$\left. \frac{(P^{(2)}(x))^{i_1}\cdots(P^{(m)}(x))^{i_{m-1}}}{(P^{(1)}(x))^{1+2i_1+\cdots+mi_{m-1}}} \right)$$

Taking $x = 0$, we get:

$$\left.\frac{d^{(n_0+1)} f_P(y)}{dy^{n_0+1}}\right|_{y=a_0}$$

$$= \sum_{j=1}^{n_0} \sum_{\substack{i_1+\cdots+i_{m-1}=j \\ i_1+2i_2\cdots+(m-1)i_{m-1}=n_0}} (-1)^j \frac{(2i_1+\cdots+mi_{m-1})!}{i_1!\cdots i_{m-1}!} \frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{1+2i_1+\cdots+mi_{m-1}}}$$

Lemma 4 *Under the hypothesis of Lemma 3, the $(n_0+1)-th$ derivative of f_P with respect to y, valued at $y = P(0) = a_0$, it also can be obtained as:*

$$\sum_{k=1}^{n_0}\left(\sum_{\substack{i_1+\cdots+i_{m-1}=k\\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}} (-1)^k H_{\overline{X}}^{\overline{i}}(\overline{0}) \right. \tag{10}$$

$$\left. \frac{(1+i_1+2i_2+\cdots+(m-1)i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}\frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{1+2i_1+\cdots+mi_{m-1}}}\right)$$

Proof. Differentiating (n_0+1) times (4) with respect to y, one gets:

$$\frac{df_P^{(n_0+1)}(y)}{dy} \tag{11}$$

$$= \frac{a_0-y}{-a_1}H_y^{(n_0+1)}(X_1(y),\cdots,X_{m-1}(y)) \quad + \quad (n_0+1)\frac{1}{a_1}H_y^{(n_0)}(X_1(y),\cdots,X_{m-1}(y))$$

The n_0-th derivative of H with respect to y is given by:

$$H_y^{(n_0)}\left(X_1(y),\cdots,X_{m-1}(y)\right) \tag{12}$$

$$= \sum_{k=1}^{n_0}\sum_{i_1+\cdots+i_{m-1}=k} H_{\overline{X}}^{\overline{i}}\Big(X_1(y),\cdots,X_{m-1}(y)\Big)A_{i_1+\cdots+i_{m-1}}(y)$$

where:

$$A_{i_1+\cdots+i_{m-1}}(y)$$

$$= \sum_{\substack{n_1^1+\cdots+n_1^{n_0}=i_1\\ \cdots\cdots\cdots\cdots\\ n_{m-1}^1+\cdots+n_{m-1}^{n_0}=i_{m-1}\\ i_1^1n_1^1+\cdots+i_{m-1}^{n_0}n_{m-1}^{n_0}=n_0}} \frac{n_0!}{n_1^1! \quad i_1^1!^{n_1^1}\cdots n_{m-1}^{n_0}! \quad i_{m-1}^{n_0}!^{n_{m-1}^{n_0}}}\left(X_1^{(i_1^1)}(y)\right)^{n_1^1}\cdots\left(X_{m-1}^{(i_{m-1}^{n_0})}(y)\right)^{n_{m-1}^{n_0}}$$

$X_i(y)$ being functions (6) and $\left(X_i^{(i_i^j)}(y)\right)^{n_i^j}$, the i_i^j-th derivative of $X_i(y)$ to the power n_i^j.

Notice that the only derivatives of $X_1(y),\cdots,X_{m-1}(y)$, that do not vanish at $y=a_0$ are:

$$X_1'(a_0) = \frac{-a_2}{(-a_1)^2},\ X_2^{(2)}(a_0) = \frac{2!a_3}{(-a_1)^3},\ldots,X_{m-1}^{(m-1)}(a_0) = \frac{(-1)^{m-1}(m-1)!a_m}{(-a_1)^m} \tag{13}$$

therefore, if we make $y=a_0$ in (12), the superscripts take the values:

$$i_1^1=1, i_1^2=0,\cdots,i_1^{n_0}=0 \quad i_2^1=0, i_2^2=2,\cdots,i_2^{n_0}=0,\ldots,i_{m-1}^1=0,\cdots,i_{m-1}^{m-1}=m-1$$

and

$$n_1^1=i_1, n_1^2=0,\cdots,n_1^{n_0}=0, \quad n_2^1=0, n_2^2=i_2,\cdots,n_2^{n_0}=0,\ldots,n_{m-1}^1=0,\cdots,i_{m-1}^{m-1}=i_{m-1}$$

so (12) becomes:

$$H_y^{(n_0)}\left(X_1(a_0),\cdots,X_{m-1}(a_0)\right)=\sum_{k=1}^{n_0}\left(\sum_{\substack{i_1+\cdots+i_{m-1}=k\\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}} H_{\overline{X}}^{\overline{i}}(\overline{0})\right. \tag{14}$$

$$\left.\frac{(i_1+2i_2+\cdots+(m-1)i_{m+1})!}{i_1!i_2!2!^{i_2}\cdots i_{m-1}!(m-1)!^{i_{m-1}}}(X_1^{(1)}(a_0))^{i_1}\ldots(X_{m-1}^{(m-1)}(a_0))^{i_{m-1}}\right)$$

Substituting (13) in (14), we arrive at:

$$H_y^{(n_0)}\left(X_1(a_0),\cdots,X_{m-1}(a_0)\right)$$

$$=\sum_{k=1}^{n_0}\sum_{\substack{i_1+\cdots+i_{m-1}=k\\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}}(-1)^k H_{\overline{X}}^{\overline{i}}(\overline{0})\frac{(i_1+2i_2+\cdots+(m-1)i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}\frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{2i_1+\cdots+mi_{m-1}}}$$

According to (11), this implies that the $(n_0+1)-th$ derivative of f_P with respect to y, valued at $y=a_0$, can be written as:

$$\sum_{k=1}^{n_0}\sum_{\substack{i_1+\cdots+i_{m-1}=k\\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}}(-1)^k H_{\overline{X}}^{\overline{i}}(\overline{0})\frac{(1+i_1+2i_2+\cdots+(m-1)i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}\frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{1+2i_1+\cdots+mi_{m-1}}}$$

Theorem 1 *Let $j_1,j_2,\cdots,j_{m-1}$ be $(m-1)$ non negative integer numbers, so that $p=j_1+\cdots+j_{m-1}$. Then there is H, a solution of (2), whose $p-th$ partial derivative with respect to x_1, j_1 times,...and with respect to x_{m-1}, j_{m-1} times, valued at $\overline{0}=(0,\cdots,0)$ is:*

$$H_{X_1,\cdots,X_{m-1}}^{(j_1,\cdots,j_{m-1})}(\overline{0})=\frac{(2j_1+\cdots+mj_{m-1})!}{(1+j_1+\cdots+(m-1)j_{m-1})!} \tag{15}$$

And the coefficients of Taylor series of H around the origin are:

$$d(i_1,\ldots,i_{m-1})=\frac{(2j_1+\cdots+mj_{m-1})!}{i_1!\ldots i_{m-1}!(1+j_1+\cdots+(m-1)j_{m-1})!} \tag{16}$$

Proof. Given $j_1,\ldots,j_{m-1}$ by hypothesis, we choose $n_0=j_1+2j_2+\cdots+(m-1)j_{m-1}$, then (8) equals (10) because, according to Lemmas 3 and 4, both expressions coincide with the (n_0+1)-th derivative of $f_P(y)$, at $y=a_0$, arriving at:

$$\begin{aligned}0 &= \sum_{k=1}^{n_0}\left(\sum_{\substack{i_1+\cdots+i_{m-1}=k\\ i_1+2i_2+\cdots+(m-1)i_{m-1}=n_0}}\left(H_{\overline{X}}^{\overline{i}}(\overline{0})\frac{(1+i_1+2i_2+\cdots+(m-1)i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}\right.\right.\\ &- \left.\left.\frac{(2i_1+\cdots+mi_{m-1})!}{i_1!\cdots i_{m-1}!}\right).(-1)^k\frac{a_2^{i_1}\cdots a_m^{i_{m-1}}}{a_1^{1+2i_1+\cdots+mi_{m-1}}}\right)\end{aligned}$$

therefore, as $a_0, a_1, \ldots, a_m$ are arbitrary real numbers, we obtain:

$$H_{\overline{X}}^{\overline{i}}(\overline{0}) = \frac{(2i_1 + \cdots + mi_{m-1})!}{(1 + i_1 + 2i_2 + \cdots + (m-1)i_{m-1})!}$$

for all $(i_1, \cdots, i_{m-1})$ so that $i_1 + \cdots + i_{m-1} = k$ and $i_1 + 2i_2 + \cdots + (m-1)i_{m-1} = n_0$, with $k = 1, \cdots, n_0$. If we take $k = p$ and $i_1 = j_1, \cdots, i_{m-1} = j_{m-1}$, the result is established.

Lemma 5 *Let m be an integer number so that $m \geq 2$, then the inequality:*

$$d(i_1, \cdots, i_{m-1}) \leq \frac{n!}{i_1! \cdots i_{m-1}!} \frac{1}{T} \frac{1}{n} T^n, \quad \forall\, n \geq 1, \quad \forall\, m \geq 2 \tag{17}$$

holds, with $i_1 + \cdots + i_{m-1} = n$, and $T = \frac{m^m}{(m-1)^{m-1}}$.

Proof. Given the polynomials:

$$P_1(x) = m(mx + m - 1) \ldots (mx + 1) \quad \forall x \geq 0 \text{ and}$$
$$Q_1(x+1) = ((m-1)x + m) \ldots ((m-1)x + 2) \quad \forall x \geq 0$$

it is easy to show that:

$$f(x) = \frac{P_1(x)(x+1)}{Q_1(x+1)x}$$

verifies:

1. If $m = 2$:
   (a) Is increasing in $[4,\infty)$.
   (b) Its restriction to integers: $\{f(n)\}_{n \geq 4}$ is an increasing sequence.
2. If $m \geq 3$:
   (a) Is increasing in $[3,\infty)$.
   (b) Its restriction to integers: $\{f(n)\}_{n \geq 3}$ is an increasing sequence.
3. The sequence $f(n)$ converges to T as $n \to \infty$.

Therefore, if $m = 2$, $\quad f(n) \leq T \quad \forall n \geq 4$, if $m \geq 3$, $\quad f(n) \leq T \quad \forall n \geq 3$.

We start by proving (17) in the case $m = 2$. For $n = 1, 2, 3$, (17) holds. For $n = k$, $k > 3$ we have that:

$$\frac{d(k+1)(k+1)}{d(k)k} = \frac{P_1(k)(k+1)}{Q_1(k+1)k} = f(k) \leq T$$

Hence:

$$d(k+1) \leq T d(k) \frac{k}{k+1}$$

By induction hypothesis the result is concluded.

We continue with the case $m \geq 3$. Our plan is:

1. In the first place we are going to prove that:
$$d(0,\ldots,n) \leq \frac{1}{n}T^{n-1}, \quad \forall\, n \geq 1 \tag{18}$$
Notice that the number of parameters of d is $m-1$.

2. In the second place, using (18), (17) will be shown.

1. For $m \geq 3$, the inequalities:
$$f(1) = 2m \quad \leq \quad f(2) = \frac{9m-3}{4} \quad \leq \quad f(3) = \frac{4(4m-2)(4m-1)}{3(3m-1)}$$
are satisfied. Then $\{f(n)\}_{n\geq 1}$ is increasing. For $n=1$, $d(0,\ldots,1)$ satisfies (17). For $n=k$, $k>1$ we have that:
$$\frac{d(0,\ldots,k+1)(k+1)}{d(0,\ldots,k)k} = \frac{P_1(k)(k+1)}{Q_1(k+1)k} = f(k) \leq T$$
Hence:
$$d(0,\ldots,k+1) \leq d(0,\ldots,k)\frac{k}{k+1}T$$
Therefore (18) follows by induction hypothesis.
2. For $n=0$ (17) is obvious. For $n>0$:
$$\begin{aligned} d(0,\ldots,0,n)n! &= \frac{(mn)!}{((m-1)n+1)!} \\ &= \frac{(mi_1+mi_2+\cdots+mi_{m-1})!}{((m-1)i_1+(m-1)i_2+\cdots+(m-1)i_{m-1}+1)!} \\ &= \frac{mn}{(m-1)n+1}\frac{mn-1}{(m-1)n}\cdots\frac{(2i_1+3i_2+\cdots+mi_{m-1})!}{(i_1+2i_2+\cdots+(m-1)i_{m-1}+1)!} \end{aligned} \tag{19}$$
Notice that the last term of product (19) is equal to $d(i_1,\ldots,i_{m-1})i_1!\ldots i_{m-1}!$ As:
$$\frac{2i_1+3i_2+\cdots+mi_{m-1}+K}{i_1+2i_2+\cdots+(m-1)i_{m-1}+1+K} \geq 1$$
with $i_1+\cdots+i_{m-1}=n\geq 1$, for all integer number $K\geq 1$, then all the terms of product (19) are greater than or equal to 1, so the proof is completed.

Theorem 2 *Let $H:\mathcal{R}^{m-1}\to\mathcal{R}$ be the function:*
$$H(x) = \sum_{n=0}^{\infty}\sum_{i_1+\cdots+i_{m-1}=n} d(i_1,\cdots,i_{m-1})x_1^{i_1}\cdots x_{m-1}^{i_{m-1}}$$
where $d(i_1,\cdots,i_{m-1})$ are coefficients (16). Then H converges uniformly in:
$$D_H = \left\{(x_1,\cdots,x_{m-1})\in\mathcal{R}^{m-1}, |x_1|+|x_2|+\cdots+|x_{m-1}| \leq \frac{1}{T}\right\} \tag{20}$$
where T was introduced in (17). Futhermore H is a solution of (2).

Proof. If we consider the absolute value of the series H, then from Lemma 5 we arrive at:

$$|H(x)| \leq \sum_{n=0}^{\infty}(|x_1| + \cdots + |x_{m-1}|)^n d(0, \ldots, n) \leq \sum_{n=0}^{\infty}(|x_1| + \cdots + |x_{m-1}|)^n T^n$$

From the second inequality uniform convergence is followed in the case $|x_1| + \cdots + |x_{m-1}| < 1/T$. On the other hand if $|x_1| + \cdots + |x_{m-1}| = 1/T$ then uniform convergence is verified, from the first inequality by Raabe Criterion. The fact that H is a solution of (2) is follows from Lemma 1 and Theorem 1.

Theorem 3 *Given polynomial function $P(x)$, according to (3), then function (4), $f_P(y)$, satisfies:*

1. *It is uniformly convergent in:*

$$D_{f_P} = \left\{ y \in \mathcal{R}; \frac{|a_0 - y||a_2|}{|a_1|^2} + \cdots + \frac{|a_0 - y|^{m-1}|a_m|}{|a_1|^m} \leq \frac{(m-1)^{m-1}}{m^m} = \frac{1}{T} \right\}$$

2. *It is the inverse of $P(x)$ around $y = a_0$.*

3. *If $0 \in D_{f_P}$, then the series:*

$$f_P(0) = \frac{a_0}{-a_1} \sum_{n=0}^{\infty} \sum_{i_1+\cdots+i_{m-1}=n} d(i_1, \cdots, i_{m-1}) \left(\frac{a_0 a_2}{(-a_1)^2} \right)^{i_1} \cdots \left(\frac{a_0^{m-1} a_m}{(-a_1)^m} \right)^{i_{m-1}}$$

converges to r, either the positive root of $P(x)$ or the negative root of $P(x)$ closest to the origin.

Proof. 1. It is immediate because H is uniformly convergent in domain (20).
2. In Lemma 2 was seen that $f_P(y)$ is the inverse of $P(x)$. On the other hand, consider the function:

$$S(y) = \frac{|a_0 - y||a_2|}{|a_1|^2} + \cdots + \frac{|a_0 - y|^{m-1}|a_m|}{|a_1|^m}$$

with $a_1 \neq 0$. Notice that $S(y)$ is greater than or equal to zero, and it approaches zero as y goes to a_0, so, by its continuity at $y = a_0$, D_{f_P} contains an open ball with center at $y = a_0$, therefore $f_P(y)$ is well defined around a_0.
3. It is followed from the continuity of f_P.

Let us give an example to illustrate these ideas.

Example

Consider the polynomial equation $P(x) = x^5 - 2x^4 + 2x^3 + x^2 - 6x + 1 = 0$, then from (4) the series:

$$\frac{1-y}{6}\sum_{n=0}^{\infty}\left(\sum_{i_1+i_2+i_3+i_4=n}\frac{(2i_1+3i_2+4i_3+5i_4)!}{i_1!i_2!i_3!i_4!(1+i_1+2i_2+3i_3+4i_4)!}\right.$$
$$\left.\left(\frac{(1-y)}{(-6)^2}\right)^{i_1}\left(\frac{(1-y)^2\,2}{(-6)^3}\right)^{i_2}\left(\frac{(1-y)^3\,(-2)}{(-6)^4}\right)^{i_3}\left(\frac{(1-y)^4}{(-6)^5}\right)^{i_4}\right)$$

is the inverse of $y = P(x)$ in the region:

$$D_{f_P} = \left\{y \in \mathcal{R};\ \frac{|1-y|}{6^2} + \frac{|1-y|^2\,2}{6^3} + \frac{|1-y|^3\,2}{6^4} + \frac{|1-y|^4\,2}{6^5} \leq \frac{4^4}{5^5}\right\}.$$

As zero belongs to D_{f_P}, then the closest positive root to the origin is $r = f_P(0)$.

3 Inverse function around any non-singular point

Suppose that there is $u \in \mathcal{R}$, with $P'(u) \neq 0$, so that $f_P(P(u))$ is not well defined. In this section we deal with this case.

Theorem 4 *Given polynomial equation (3) and $u \in \mathcal{R}$ with $P'(u) \neq 0$, we consider $P_u(x) = P(u+x)$, then the function:*

$$f_{P_u}(y) = \frac{P(u)-y}{-P^{(1)}(u)} H\left(Y_1(u,y), \ldots, Y_{m-1}(u,y)\right) \tag{21}$$

defined in:

$$D_{f_{P_u}} = \{(u,y) \in \mathcal{R}^2,\ |Y_1(u,y)| + \cdots + |Y_{m-1}(u,y)| \leq 1/T\}$$

is the inverse of $P_u(x)$ around $y = P(u)$, where:

$$Y_i(u,y) = \frac{(P(u)-y)^i P^{(i+1)}(u)}{(i+1)!(-P^{(1)}(u))^{i+1}} \quad 1 \leq i \leq m-1$$

Proof. Let us write the Taylor's Formula of $P(x)$ around $x = u$:

$$P(y) = P(u) + P^{(1)}(u)(y-u) + \frac{P^{(2)}(u)}{2!}(y-u)^2 + \cdots + \frac{P^{(m)}(u)}{m!}(y-u)^m \tag{22}$$

Changing $(y-u)$ by x in (22), we have:

$$P_u(x) = P(x+u) = P(u) + P^{(1)}(u)x + \frac{P^{(2)}(u)}{2!}x^2 + \cdots + \frac{P^{(m)}(u)}{m!}x^m$$

And the result is followed from applying Theorem 3 to $P_u(x)$.

Corollary 1 *With the same hypothesis as Theorem 4, let F_P be the function:*

$$F_P(y) = u + f_{P_u}(y) \tag{23}$$

with $(u, y) \in D_{f_{P_u}}$, then (23) is the inverse of $P(x)$ around $x = u$.

Proof. Consider $y = P(u_1)$ with $(u_1, y) \in D_{f_{P_u}}$, then:

$$\begin{aligned} F_P(y) &= u + f_{P_u}(y) = u + f_{P_u}(P(u_1)) \\ &= u + f_{P_u}(P_u(u_1 - u)) = u + u_1 - u = u_1 \end{aligned}$$

Notice that if $u = 0$, then $F_P(y) = f_P(y)$.

4 A comparison with the quadratic and cubic equation

In this section we illustrate the relation of quadratic and cubic equations to the functions $f_P(y)$ introduced in Theorem 3.

Consider a quadratic equation $x^2 + bx + c - y = 0$, with $c, b \in \mathcal{R}$. By solving for x we get two functions:

$$\begin{aligned} x &= S_1(y) = \frac{-b + \sqrt{b^2 - 4(c-y)}}{2} \\ x &= S_2(y) = \frac{-b - \sqrt{b^2 - 4(c-y)}}{2} \end{aligned} \tag{24}$$

According to Theorem 3, a inverse function of $y = P(x) = x^2 + bx + c$ is:

$$f_P(y) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!(-b)^{2n+1}} (c-y)^{n+1} \tag{25}$$

Then it is easy to prove that $f_P(y)$ is the Taylor series of $S_1(y)$.

Now we consider the cubic equation $x^3 + px + q - y = 0$, with $p,\ q \in \mathcal{R}$. By using Cardan-Tartaglia formula we get three functions (one for each solution of $x^3 + px + q - y = 0$), given by the expression:

$$T(y) = \sqrt[3]{-\frac{q-y}{2} + \sqrt{\frac{(q-y)^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q-y}{2} - \sqrt{\frac{(q-y)^2}{4} + \frac{p^3}{27}}}$$

Then a inverse function of $y = P(x) = x^3 + px + q$, provides by Theorem 3 is:

$$f_P(y) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n+1)!(-p)^{3n+1}} (q-y)^{2n+1} \tag{26}$$

that is the Taylor series of one of the branches of $T(y)$, the closest to the origin.

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