Inverse Functions of Polynomials Around All Its Roots

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Abstract

In [2] inverse functions of polynomials, from a theoretical point of view were built. In other words, given a polynomial function $y = P(x) = a_0 + a_1x + \cdots + a_mx^m$, with $a_i \in \mathcal{R},\ 0 \leq i \leq m$, and a real number u so that $P'(u) \neq 0$, we got an explicit function $F_P(y)$ that satisfies $x = F_P(P(x))$ around $x = u$.

Now, in this paper, above results are generalized with the aim of building the inverse functions of $P(x)$ around all its roots.

On the other hand, as a potencial application, we are exploring the possibilities of the functions $F_P(y)$ in order to find all the roots of polynomial equations and algebraic systems.

Keywords: Polynomial functions, quadratic formula, Cardan-Tartaglia Formula, Galois Theorem, Adomian Method, Decomposition Method, inverse of polynomials.

1 Introduction

Polynomial equation:

$$0 = a_0 + a_1x + \cdots + a_mx^m \tag{1}$$

arises in many fields of sciences, engineering, economy and statistics. In such areas available algorithms for solving (1), ([1], are effective for the "average"polynomial of a small or moderate degree, but they are heuristic in general and do not converge to the roots of (1) unless some suitable initial values are given. Furthermore no global procedures are provided in order to find such a convenient approximation, for a general polynomial equation.

When the degree, m, grows beyond 10 or 20, the present-day necessities for solving (1) become more and more extended, but above all in the important case of *Computer Algebra*, where solving (1) for a larger m is needed. Here is where equation (1) keeps playing its role nowadays, both as a research problem and as a part of computing tasks.

As it was expected, all these problems are increased in the case of polynomial systems.

As a potencial application, one explores the possibilities of inverse functions of polynomials, $F_P(y)$, introduced in this paper, in order to solve the problems of convergence and initial values, that we have outlined above, for both cases: polynomial equations and polynomial systems.

This article is organized as follows: In section 2 we summarize some recent results of [2] that will be used in the following. Section 3 deals with constructing inverse functions of polynomials around all its roots. Section 4 provides some examples in order to illustrate the main ideas.

Throughout this paper all the sets of subscripts or superscripts will be considered as non-negative integer numbers; the degree of $P(x)$ will be denoted by m; and the partial derivative of H with respect to X_1, q_1 times, $\ldots$, and with respect to X_{m-1}, q_{m-1} times, valued at zero, by $H^{q_1,\ldots,q_{m-1}}_{X_1,\ldots,X_{m-1}}(\overline{0})$.

The proof of the non-existence of solutions of (1) by radicals, for a degree greater or equal to five, extended the belief in the non-existence of no-one else explicit formulae, giving up this line of research until present-day. But the functions $f_P^{(j,i)}(y)$, which we introduce in this paper, show that it might be possible, as it is illustrated in the examples of Section 4. Furthermore, in the case of the polynomial equation $x^m - a_0 = 0$, $f_P^{(j,i)}(0)$ coincides with the usual radical $\sqrt[m]{a_0}$.

2 Previous results

Let $P(x)$ be the polynomial function:

$$y = P(x) = a_0 + a_1 x + \cdots + a_m x^m \tag{2}$$

where $a_0, a_1, \cdots, a_m$ are real numbers with $a_1, a_m \neq 0$. With the purpose of finding a inverse function of (2), denoted by $f_P(y)$, around $y = a_0$, we begun, in [2], by defining and solving the functional equation:

$$h(x) = 1 + x_1 h^2(x) + x_2 h^3(x) + \cdots + x_{m-1} h^m(x) \tag{3}$$

where $m \geq 2$ is an integer number, $x = (x_1, \cdots, x_{m-1})$ is a vector of $\mathcal{R}^{m-1}$ and the unknown, $h : \mathcal{R}^{m-1} \to \mathcal{C}$, is a function with $h(0, \cdots, 0) = 1$. From a solution, H, of (3), we gave $f_P(y)$ as:

$$f_P(y) = k(y) H(X_1(y), \cdots, X_{m-1}(y)) \tag{4}$$

where the functions $X_i(y)$, $1 \leq i \leq m-1$, were given by:

$$X_i(y) = \frac{(a_0 - y)^i a_{i+1}}{(-a_1)^{i+1}} \quad \text{and} \quad k(y) = \frac{a_0 - y}{-a_1} \tag{5}$$

In order to build $f_P(y)$ in an explicit way, the following results were proved in [2]:

1. Its Taylor series was defined as:

$$f_P(y) = \frac{a_0 - y}{-a_1} \sum_{n=0}^{\infty} \sum_{q_1 + \cdots + q_{m-1} = n} d(q_1, \cdots, q_{m-1}) \left(\frac{(a_0 - y) a_2}{(-a_1)^2} \right)^{q_1} \cdots \left(\frac{(a_0 - y)^{m-1} a_m}{(-a_1)^m} \right)^{q_{m-1}} \tag{6}$$

with:

$$d(q_1,\ldots,q_{m-1}) = \frac{(2q_1+\cdots+mq_{m-1})!}{q_1!\ldots q_{m-1}!(1+q_1+\cdots+(m-1)q_{m-1})!} \tag{7}$$

2. (6) is uniformly convergent in:

$$D_{f_P} = \left\{ y \in \mathcal{R};\ \frac{|a_0-y||a_2|}{|a_1|^2} + \cdots + \frac{|a_0-y|^{m-1}|a_m|}{|a_1|^m} \leq \frac{(m-1)^{m-1}}{m^m} \right\} \tag{8}$$

3. It is satisfied that $x = f_P(P(x))$, around $x = 0$.

4. If $0 \in D_{f_P}$, then the series:

$$f_P(0) = \frac{a_0}{-a_1}\sum_{n=0}^{\infty}\sum_{q_1+\cdots+q_{m-1}=n} d(q_1,\cdots,q_{m-1})\left(\frac{a_0a_2}{(-a_1)^2}\right)^{q_1}\cdots\left(\frac{a_0^{m-1}a_m}{(-a_1)^m}\right)^{q_{m-1}} \tag{9}$$

converges to r, either the positive root of $P(x)$ or the negative root of $P(x)$ closest to the origin.

Finally the Taylor series of the inverse function of $P(x)$, around any non-singular point, $y = P(u)$ with $P'(u)$ different to zero, was obtained. In fact, given polynomial function (2) and $u \in \mathcal{R}$ with $P'(u) \neq 0$, one considers $P_u(x) = P(u+x)$, then the function:

$$F_P(u) = u + f_{P_u}(y) = \frac{P(u)-y}{-P^{(1)}(u)} H\left(Y_1(u,y),\ldots,Y_{m-1}(u,y)\right) \tag{10}$$

defined in:

$$D_{F_P} = \{(u,y) \in \mathcal{R}^2,\ |Y_1(u,y)| + \cdots + |Y_{m-1}(u,y)| \leq \frac{(m-1)^{m-1}}{m^m}\}$$

it is the inverse of $P(x)$ around $y = P(u)$, where:

$$Y_i(u,y) = \frac{(P(u)-y)^i P^{(i+1)}(u)}{(i+1)!(-P^{(1)}(u))^{i+1}}, \quad 1 \leq i \leq m-1$$

3 Inverse function of P(x) around its roots

Once the main results of [2] have been recalled, observe that functions (6) and (10) have important restrictions:

1. (6) and (10) can only be defined around the reals root of $P(x)$, so its complex roots can not be computed as $f_P(0)$ or $F_P(0)$.

2. On the other hand the real numbers u in $F_P(y) = u + f_{P_u}(y)$, in order to find all the real roots of $P(x)$, are not given in an explicit way.

The next examples illustrate these difficulties.

Example 1 *Consider the polynomial equation $P(x) = x^5 + 9x^4 + 33x^3 + 61x^2 + 46x + 1 = 0$, then from (6) the series:*

$$\frac{1-y}{-46}\sum_{n=0}^{\infty}\sum_{q_1+q_2+q_3+q_4=n} d(q_1,\ldots,q_{m-1})\left(\frac{(1-y)\,61}{(-46)^2}\right)^{q_1}\left(\frac{(1-y)^2\,33}{(-46)^3}\right)^{q_2}\left(\frac{(1-y)^3\,9}{(-46)^4}\right)^{q_3}\left(\frac{(1-y)^4}{(-46)^5}\right)^{q_4}$$

where:

$$d(q_1,\dots,q_4)=\frac{(2q_1+3q_2+4q_3+5q_4)!}{q_1!q_2!q_3!q_4!(1+q_1+2q_2+3q_3+4q_4)!}$$

it is the inverse of $y=P(x)$ *in the region:*

$$D_{f_P}=\left\{y\in\mathcal{R};\frac{|1-y|\;61}{46^2}+\frac{|1-y|^2\;33}{46^3}+\frac{|1-y|^3\;9}{46^4}+\frac{|1-y|^4}{46^5}\le\frac{4^4}{5^5}\right\}$$

As zero belongs to D_{f_P}*, then the closest positive root to the origin is* $r=f_P(0)$.

Example 2 *Consider again the polynomial equation* $P(x)=x^5+9x^4+33x^3+61x^2+46x+1=0$ *of Example 1. There, we have found its closet root to the origin. Now, using (10), we are going to compute another of its real roots. By making the change of variable* $x=z-2$*, we obtain:*

$$P_{-2}(z)=P(z-2)=z^5-z^4+z^3-z^2-10z+1=0$$

Then from (10) the series:

$$-2+\frac{1-y}{10}\sum_{n=0}^{\infty}\sum_{q_1+q_2+q_3+q_4=n}d(q_1,\dots,q_4)\left(\frac{(1-y)\,(-1)}{10^2}\right)^{q_1}\left(\frac{(1-y)^2}{10^3}\right)^{q_2}\left(\frac{(1-y)^3\,(-1)}{10^4}\right)^{q_3}\left(\frac{(1-y)^4}{10^5}\right)^{q_4}$$

where $d(q_1,\dots,q_4)$ *is the same as in the preceding example, it is the inverse of* $y=P(x)$ *in the region:*

$$D_{F_P}=\left\{y\in\mathcal{R};\frac{|1-y|}{10^2}+\frac{|1-y|^2}{10^3}+\frac{|1-y|^3}{10^4}+\frac{|1-y|^4}{10^5}\le\frac{4^4}{5^5}\right\}$$

As zero belongs to D_{F_P}*, then* $r=F_P(0)$ *is other of the real roots of* $P(x)$.

In this section we accomplish the task of dealing with such deficiencies. The proposed solution is to build, in an explicit way, one inverse function of $P(x)$, $(f_P)_v(y)$, $1\le v\le m$, for each one of its roots, r_v, in such a manner as $r_v=(f_P)_v(0)$. In order to achieve this goal, we begin with Definition 1, which introduces a set, F, of functional equations.

Hereafter j and i will denote integer numbers, so that $m\ge j>i\ge 0$.

Definition 1 *Let* $x=(x_1,\cdots,x_{m-1})$ *be a vector of* $\mathcal{R}^{m-1}$ *and* $h:\mathcal{R}^{m-1}\to\mathcal{C}$ *a function. Then we define the set,* F*, of functional equations, with* $\binom{m+1}{2}$ *elements, given by:*

$$\begin{aligned}F=\{h^j(x)&=x_1h^0(x)+x_2h(x)+\cdots+x_ih^{i-1}(x)+h^i(x)+x_{i+1}h^{i+1}(x)+\dots\\&+x_{j-1}h^{j-1}(x)+x_jh^{j+1}(x)+\cdots+x_{m-1}h^m(x),\ h(0,\dots,0)=1;\ m\ge j>i\ge 0\}\end{aligned}\tag{11}$$

h being the unknown function. The element of F*, corresponding to the parameters* j *and* i*, will be denoted by* (j,i)*-functional equation.*

The proof of Theorem 1 is not included here because, once adapted to the new hypotheses, it is very similar to the proof of Lemma 1 introduced in [2].

Theorem 1 *Each (j,i)-functional equation of the set F is solvable.*

From each solution, $H^{(j,i)}$, of each (j,i)-functional equation of F, Theorem 2 provides $(j-i)$ inverse functions of (12).

Theorem 2 *Fixed m, j and i, consider the polynomial function:*

$$y = P(x) = a_0 + a_1 x + \cdots + a_m x^m \tag{12}$$

where $a_0, a_1, \cdots, a_m$ are real numbers with $a_j, a_m \neq 0$. Let $k_v^{(j,i)}(y)$ be the functions:

$$k_v^{(j,i)}(y) = \begin{cases} \sqrt[j]{\frac{a_0-y}{-a_j}} & \text{if, } i=0 \\ \sqrt[j-i]{\frac{a_i}{-a_j}} & \text{if, } i>0 \end{cases} \qquad 1 \leq v \leq j-i \tag{13}$$

where the subscript v denotes the $v-th$ root. Finally, let $H^{(j,i)}$ be a solution of the (j,i)-functional equation, then:

$$(f_P^{(j,i)})_v(y) = k_v^{(j,i)}(y) H^{(j,i)}\left(X_1^{(j,i)})_v(y) \ldots, (X_{m-1}^{(j,i)})_v(y)\right) \tag{14}$$

verifies $x = (f_P^{(j,i)})_v(P(x))$, where the functions $(X_p^{(j,i)})_v(y)$, $1 \leq p \leq m-1$, $1 \leq v \leq j-i$, are:

First, $i=0$:

$$\begin{array}{ll} (X_1^{(j,0)})_v(y) = \frac{a_1}{-a_j}\left(k_v^{(j,0)}(y)\right)^{1-j}, & (X_2^{(j,0)})_v(y) = \frac{a_2}{-a_j}\left(k_v^{(j,0)}(y)\right)^{2-j} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ (X_{j-1}^{(j,0)})_v(y)) = \frac{a_{j-1}}{-a_j}\left(k_v^{(j,0)}(y)\right)^{-1}, & (X_j^{(j,0)})_v(y) = \frac{a_{j+1}}{-a_j}\left(k_v^{(j,0)}(y)\right)^{1} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ (X_{m-1}^{(j,0)})_v(y) = \frac{a_m}{-a_j}\left(k_v^{(j,0)}(y)\right)^{m-j} & \end{array} \tag{15}$$

Second, $i>0$:

$$\begin{array}{ll} (X_1^{(j,i)})_v(y) = \frac{a_0-y}{-a_j}\left(k_v^{(j,i)}(y)\right)^{-j}, & (X_2^{(j,i)})_v(y) = \frac{a_1}{-a_j}\left(k_v^{(j,i)}(y)\right)^{1-j} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ (X_i^{(j,i)})_v(y) = \frac{a_{i-1}}{-a_j}\left(k_v^{(j,i)}(y)\right)^{i-1-j}, & (X_{i+1}^{(j,i)})_v(y) = \frac{a_{i+1}}{-a_j}\left(k_v^{(j,i)}(y)\right)^{i+1-j} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ (X_{j-1}^{(j,i)})_v(y)) = \frac{a_{j-1}}{-a_j}\left(k_v^{(j,i)}(y)\right)^{-1}, & (X_j^{(j,i)})_v(y) = \frac{a_{j+1}}{-a_j}\left(k_v^{(j,i)}(y)\right)^{1} \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ (X_{m-1}^{(j,i)})_v(y) = \frac{a_m}{-a_j}\left(k_v^{(j,i)}(y)\right)^{m-j} & \end{array} \tag{16}$$

Proof. In fact, as $H^{(j,i)}(\overline{X}_v^{(j,i)}(y))$, with $\overline{X}_v^{(j,i)}(y) = ((X_1^{(j,i)})_v(y), \ldots, (X_{m-1}^{(j,i)})_v(y))$, is a solution of the (j,i) - functional equation of set (11), it follows that:

$$\begin{aligned}
(-a_j)(f_P^{(j,i)})_v^j(y) &= (-a_j)(k_v^{(j,i)}(y))^j (H^{(j,i)})^j(\overline{X}_v^{(j,i)}(y)) \\
&= (-a_j)(k_v^{(j,i)}(y))^j \Big((X_1^{(j,i)})_v(y)(H^{(j,i)})^0(\overline{X}_v^{(j,i)}(y)) + \ldots \\
&+ (X_i^{(j,i)})_v(y)(H^{(j,i)})^{i-1}(\overline{X}_v^{(j,i)}(y)) + (H^{(j,i)})^i(\overline{X}_v^{(j,i)}(y)) \\
&+ (X_{i+1}^{(j,i)})_v(y)(H^{(j,i)})^{i+1}(\overline{X}_v^{(j,i)}(y)) + (X_{j-1}^{(j,i)})_v(y)(H^{(j,i)})^{j-1}(\overline{X}_v^{(j,i)}(y)) \\
&+ (X_j^{(j,i)})_v(y)(H^{(j,i)})^{j+1}(\overline{X}_v^{(j,i)}(y)) + \cdots + (X_{m-1}^{(j,i)})_v(y)(H^{(j,i)})^m(\overline{X}_v^{(j,i)}(y)) \Big)
\end{aligned} \tag{17}$$

Substituting functions (15) or (16) in (17) one gets:

$$y = a_0 + a_1(f_P^{(j,i)})_v(y) + a_2(f_P^{(j,i)})_v^2(y) + \cdots + a_m(f_P^{(j,i)})_v^m(y)$$

Therefore $x = (f_P^{(j,i)})_v(y) = (f_P^{(j,i)})_v(P(x))$ and the result holds.
In order to obtain the functions $(f_P^{(j,i)}(y))_v$, $1 \le v \le j-i$, in an analytic way, Theorem 3 computes the coefficients of the Taylor series of $H^{(j,i)}$, $d^{(j,i)}$, that are introduced in (20). Theorem 4 shows that $H^{(j,i)}$ is analytic in domain (52). Finally, Theorem 5 gives $(f_P^{(j,i)}(y))_v$, $1 \le v \le j-i$, in an explicit way, by defining its series in (53).

Theorem 3 *Let $q_1, q_2, \cdots, q_{m-1}$ and $n \ge 1$ be non negative integer numbers, so that $n = q_1 + \cdots + q_{m-1}$ and, the rational number:*

$$\begin{aligned}
W_{q_1,\ldots,q_{m-1}}^{(j,i)} &= \frac{(0-i)q_1 + (1-i)q_2 + \cdots + (i-1-i)q_i + (i+1-i)q_{i+1} + \ldots}{j-i} \\
&+ \frac{(j-1-i)q_{j-1} + (j+1-i)q_j + \cdots + (m-i)q_{m-1} + 1}{j-i} - 1
\end{aligned} \tag{18}$$

Then, there is $H^{(j,i)}$, a solution of the (j,i)-functional equation of set (11), whose $n-th$ partial derivative with respect to x_1, q_1 times, ..., and with respect to x_{m-1}, q_{m-1} times, valued at $\overline{0} = (0, \cdots, 0)$, is:

$$(H^{(j,i)})^{(q_1,\cdots,q_{m-1})}_{X_1^{(j,i)},\cdots,X_{m-1}^{(j,i)}}(\overline{0}) = \begin{cases} \frac{1}{j-i};\ n=1 \\ \frac{1}{j-i} W_{q_1,\ldots,q_{m-1}}^{(j,i)} \Big(W_{q_1,\ldots,q_{m-1}}^{(j,i)} - 1\Big) \ldots \Big(W_{q_1,\ldots,q_{m-1}}^{(j,i)} - (n-2)\Big);\ n \ge 2 \end{cases} \tag{19}$$

And the coefficients of the Taylor series of $H^{(j,i)}$ around the origin, for $n \ge 1$, are:

$$d^{(j,i)}(q_1,\ldots,q_{m-1}) = \frac{1}{j-i} \frac{W_{q_1,\ldots,q_{m-1}}^{(j,i)} \Big(W_{q_1,\ldots,q_{m-1}}^{(j,i)} - 1\Big) \ldots \Big(W_{q_1,\ldots,q_{m-1}}^{(j,i)} - (n-2)\Big)}{q_1! \ldots q_{m-1}!} \tag{20}$$

Proof. In order to make the notation clearer and without loss of generality, since the reasoning, which is done next, can be generalized easily, we only are going to prove the case in which the function $H^{(j,i)}$ has two variables. In other words: Consider the (j,i) - functional equation:

$$h^j(x,y)) = h^i(x) + xh^{p_1}(x,y) + yh^{p_2}(x,y) \tag{21}$$

where p_1 and p_2 are integer numbers so that $0 \leq p_1 < p_2 \leq m = 3$, with $p_1,\ p_2 \neq i,\ j$, then $\forall\ n\ \geq 1$, with $q_1 + q_2 = n$, we are going to prove that:

$$\begin{aligned}(H^{(j,i)})^{(q_1,q_2)}_{x,y}(0,0) =& \frac{1}{j-i}\left(\frac{(p_1-i)q_1+(p_2-i)q_2+1}{j-i}-1\right)\dots\\ &\left(\frac{(p_1-i)q_1+(p_2-i)q_2+1}{j-i}-(n-1)\right)\end{aligned} \tag{22}$$

In fact, for $n = 1$, by derivative rules it is verified that:

$$(H^{(j,i)})^{(1,0)}_{x,y}(0,0) = \frac{1}{j-i} \quad \text{and} \quad (H^{(j,i)})^{(0,1)}_{x,y}(0,0) = \frac{1}{j-i} \tag{23}$$

For n=2:

$$\begin{aligned}(H^{(j,i)})^{(2,0)}_{x,y}(0,0) &= \frac{1}{j-i}\left(\frac{(p_1-i)2+1}{j-i}-1\right)\\ (H^{(j,i)})^{(1,1)}_{x,y}(0,0) &= \frac{1}{j-i}\left(\frac{(p_1-i)1+(p_2-i)1+1}{j-i}-1\right)\\ (H^{(j,i)})^{(0,2)}_{x,y}(0,0) &= \frac{1}{j-i}\left(\frac{(p_2-i)2+1}{j-i}-1\right)\end{aligned} \tag{24}$$

Suppose now that the result is true for $n = k$ and false for $n = k+1$, then, as the derivative rules do not depend on p_1, p_2, j and i, it will also be false, if we choose $p_1 = 2$, $p_2 = 3$, $j = 1$ and $i = 0$. From here it is followed that (7) is false. And this contradiction proves the result. With the aim of showing Theorem 4, Lemmas 1, 2, 3 and 4, that we introduce next, are needed. Lemma 1 can be easily proven.

Lemma 1 *Consider A_n and B_n given by:*

$$\begin{aligned}A_n &= \frac{1}{j-i}\left(\frac{(m-i)n+1}{j-i}-1\right)\dots\left(\frac{(m-i)n+1}{j-i}-(n-1)\right),\ \forall\, n \geq 1\\ B_n &= \frac{1}{j-i}\left(\frac{-in+1}{j-i}-1\right)\dots\left(\frac{-in+1}{j-i}-(n-1)\right),\ \forall\, n \geq 1\end{aligned} \tag{25}$$

Then:

$$A_n > 0,\ \forall n \geq 1,\ B_n \neq 0 \text{ (except in the case } i = 0 \text{ and } j = 1),\ \forall\, n \geq 1, \text{ and} \tag{26}$$

$$|A_n| \geq |B_n|,\ \forall\, n \geq 1,\ \textit{if, and only if, } m - i - j \geq 0 \tag{27}$$

Lemma 2 *From A_n and B_n, introduced in (25), let us define the sequence:*

$$S_n = max\{A_n, |B_n|\} = \begin{cases} A_n, & \textit{if } m-i-j \geq 0 \\ |B_n|, & \textit{if } m-i-j < 0 \end{cases} \tag{28}$$

$\forall\, n \geq 1$, then:

$$\left|d^{(j,i)}(q_1,\dots,q_{m-1})\right| q_1!\dots q_{n-1}! \leq S_n \tag{29}$$

where $q_1, q_2, \cdots, q_{m-1}$ are non negative integer numbers, so that $q_1 + \cdots + q_{m-1} = n$, and $d^{(j,i)}(q_1,\dots,q_{m-1})$ is given by (20).

Proof. If $d^{(j,i)} = 0$, then (29) is obvious. From (18) observe that:

$$\begin{aligned}
\frac{-in+1}{j-i} - 1 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} \le \frac{(m-i)n+1}{j-i} - 1 \\
\frac{-in+1}{j-i} - 2 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - 1 \le \frac{(m-i)n+1}{j-i} - 2 \\
\dots \quad \dots \quad \dots \\
\frac{-in+1}{j-i} - (n-1) \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) \le \frac{(m-i)n+1}{j-i} - (n-1)
\end{aligned} \tag{30}$$

Now, we distinguish two cases: First $W^{(j,i)}_{q_1,\dots,q_{m-1}} < 0$ and second $W^{(j,i)}_{q_1,\dots,q_{m-1}} > 0$. In the first one, from (26), $B_n \neq 0$, and it is concluded that:

$$\begin{aligned}
&\left| d^{(j,i)}(q_1, \cdots, q_{m-1}) \right| q_1! \dots q_{n-1}! \\
&= \frac{1}{j-i} \left| W^{(j,i)}_{q_1,\dots,q_{m-1}} \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - 1 \right) \dots \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) \right) \right| \le |B_n|
\end{aligned} \tag{31}$$

In the second case, suppose that $W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2)$ is also positive, then it is held that:

$$\begin{aligned}
&\left| d^{(j,i)}(q_1, \cdots, q_{m-1}) \right| q_1! \dots q_{n-1}! \\
&= \frac{1}{j-i} \left| W^{(j,i)}_{q_1,\dots,q_{m-1}} \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - 1 \right) \dots \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) \right) \right| \le A_n
\end{aligned} \tag{32}$$

Otherwise, suppose that $W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) < 0$, then there is k, integer, $1 \le k \le n-3$, so that $W^{(j,i)}_{q_1,\dots,q_{m-1}} - k > 0$ and $W^{(j,i)}_{q_1,\dots,q_{m-1}} - (k+1) < 0$. From here it is followed that:

$$\begin{aligned}
0 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} \le \frac{(m-i)n+1}{j-i} - 1 \\
0 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - 1 \le \frac{(m-i)n+1}{j-i} - 2 \\
\dots \quad \dots \quad \dots \quad \dots \\
0 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - k \le \frac{(m-i)n+1}{j-i} - (k+1) \\
-\frac{j-i-1}{j-i} \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - (k+1) \le \frac{j-i-1}{j-i} \le \frac{(m-i)n+1}{j-i} - (n-1) \\
-\frac{j-i-1}{j-i} - 1 \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - (k+2) \le \frac{j-i-1}{j-i} + 1 \le \frac{(m-i)n+1}{j-i} - (n-2) \\
\dots \quad \dots \quad \dots \quad \dots \\
-\frac{j-i-1}{j-i} - (n-k-3) \le W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) \le \frac{j-i-1}{j-i} + (n-k-3) \le \frac{(m-i)n+1}{j-i} - (k+2)
\end{aligned} \tag{33}$$

And then it is satisfied that:

$$\begin{aligned}
&\left| d^{(j,i)}(q_1, \cdots, q_{m-1}) \right| q_1! \dots q_{n-1}! \\
&= \frac{1}{j-i} \left| W^{(j,i)}_{q_1,\dots,q_{m-1}} \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - 1 \right) \dots \left(W^{(j,i)}_{q_1,\dots,q_{m-1}} - (n-2) \right) \right| \le A_n
\end{aligned} \tag{34}$$

Therefore:

$$\left|d^{(j,i)}(q_1,\cdots,q_{m-1})\right| q_1!\dots q_{n-1}! \leq max\{A_n,|B_n|\} = S_n \tag{35}$$

And (29) is followed.

Lemma 3 *Consider the constant:*

$$T_{(j,i)} = \begin{cases} \frac{1}{j-i}\sqrt[j-i]{\frac{(m-i)^{m-i}}{(m-j)^{m-j}}} & \textit{if } m-i-j) \geq 0 \textit{ and } j \neq m \\ 1 & \textit{if } (m-i-j) \geq 0 \textit{ and } j = m \\ \frac{1}{j-i}\sqrt[j-i]{\frac{j^j}{i^i}} & \textit{if } m-i-j < 0 \end{cases} \tag{36}$$

and the sequence:

$$M_n = \begin{cases} \frac{A_n}{n!}\frac{(n-1)!}{A_{n-1}} & \textit{if } m-i-j \geq 0 \\ \\ \frac{|B_n|}{n!}\frac{(n-1)!}{|B_{n-1}|} & \textit{if } m-i-j < 0 \end{cases} \tag{37}$$

$\forall\, n \geq j-i$*. Then:*

$$\lim_{n\to\infty} M_n = T_{(j,i)} \quad \text{and} \quad M_n \leq T_{(j,i)}, \quad \forall\, n \geq j-1 \tag{38}$$

Proof. We only are going to show the case $m-i-j \geq 0$ and $j \neq m$. The other ones can be proven following exactly the same steps. We begin by proving the first part of (38), with this aim we introduce next a new sequence, N_n, for all $n \geq j-i$, given by:

$$\begin{aligned} N_n &= \frac{A_n}{n!}\frac{(n-(j-i))!}{A_{n-(j-i)}} \\ &= \frac{(m-i)n+1-(j-i)}{(m-j)n+1-(j-i)}\cdots\frac{(m-i)n+1-(m-j)(j-i)}{(m-j)n+1-(m-j)(j-i)} \\ &\cdot\frac{(m-i)n+1-(m-j+1)(j-i)}{n}\cdots\frac{(m-i)n+1-(m-i)(j-i)}{n-(j-i-1)} \end{aligned} \tag{39}$$

and which satisfies the following proprieties:

$$1. \quad \lim_{n\to\infty} N_n = T_{(j,i)}^{j-i} \qquad 2. \quad N_n,\ \forall\, n \geq j-i, \text{ is increasing.} \tag{40}$$

In fact, the first part of (40) is obvious, and in order to prove the second one, it is proceeded as follows: One chooses the positive integer numbers $k_0, k_1, \dots k_{j-i-1}$ to satisfy the inequalities:

$$\begin{aligned} &0 \leq -1+(k_0+1)(j-i) < -1+2(j-i) < \cdots < -1+k_1(j-i) < m-i \\ &(m-i) \leq -1+(k_1+1)(j-i) < -1+(k_1+2)(j-i) < \cdots < -1+k_2(j-i) < 2(m-i) \\ &\cdots \\ &(j-i-2)(m-i) \leq -1+(k_{j-i-2}+1)(j-i) < \cdots < -1+k_{j-i-1}(j-i) < (j-i-1)(m-i) \\ &(j-i-1)(m-i) \leq -1+(k_{j-i-1}+1)(j-i) \end{aligned} \tag{41}$$

with $k_0 = 0$ and $k_{j-i-1} + 1 = m - i$. Using k_0, k_1, $\cdots$ k_{j-i-1}, we reorder the rational factors of (39) in the following manner:

$$\begin{aligned}
&\frac{(m-i)n+1-(k_0+1)(j-i)}{n}\frac{(m-i)n+1-(k_0+2)(j-i)}{(m-j)n+1-(j-i)}\cdots\frac{(m-i)n+1-k_1(j-i)}{(m-j)n+1-(k_1-1)(j-i)}\\
&\cdot\frac{(m-i)n+1-(k_1+1)(j-i)}{n-1}\frac{(m-i)n+1-(k_1+2)(j-i)}{(m-j)n+1-k_1(j-i)}\cdots\frac{(m-i)n+1-k_2(j-i)}{(m-j)n+1-(k_2-2)(j-i)}\\
&\cdots\\
&\cdot\frac{(m-i)n+1-(k_{j-i-2}+1)(j-i)}{n-(j-i-2)}\cdots\frac{(m-i)n+1-k_{j-i-1}(j-i)}{(m-j)n+1-(k_{j-i-1}-(j-i-1))(j-i)}\\
&\cdot\frac{(m-i)n+1-(k_{j-i-1}+1)(j-i)}{n-(j-i-1)}
\end{aligned} \tag{42}$$

then, each one of the rational factors of (42) is increasing in the variable n, since differentiating all of them with respect to n, we arrive at:

$$\left(\frac{(m-i)n+1-r_p(j-i)}{(m-j)n+1-(r_p-p)(j-i)}\right)' > 0 \Leftrightarrow -1 + r_p(j-i) < (m-i)p \tag{43}$$

where $1 \leq p \leq j-i-1$ and $r_p \in \{k_{p-1}+2, k_{p-1}+3, \cdots, k_p\}$, and

$$\left(\frac{(m-i)n+1-(k_p+1)(j-i)}{n-p}\right)' \geq 0 \Leftrightarrow -1 + (k_p+1)(j-i) \geq (m-i)p \tag{44}$$

where $0 \leq p \leq j-i-1$. And from inequalities (41) it is deduced that derivatives (43) are positive, and (44), non negative, so, the sequence, N_n, $\forall\, n \geq j-i$, is increasing. This done, notice that N_n can be express as:

$$N_n = \frac{A_n}{A_{n-1}n}\frac{A_{n-1}}{A_{n-2}(n-1)}\cdots\frac{A_{n-(j-i)+1}}{A_{n-j-i}(n-(j-i))} \tag{45}$$

and therefore:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty}\frac{A_n}{A_{n-1}n} = \lim_{n\to\infty}\sqrt[j-i]{N_n} = T_{(j,i)} \tag{46}$$

From (45) and (46) it is deduced the second part of (38), since N_n increasing being, if there were n_0 so that $M_{n_0} > T_{(j,i)}$, then it would be $M_{n_0} < M_{n_0+n(j-i)} < T_{(j,i)}$, $\forall\, n \geq 1$. And the proof is finished.

Lemma 4 *It is satisfied that:*

$$S_n \leq T^n_{(j,i)} n!, \quad \forall\, n \geq j-i \tag{47}$$

S_n was introduced in (28) and $T_{(j,i)}$ in (36).

Proof. According to the definition of S_n, three cases are distinguished:

1. $S_n = A_n$, with $m-j-i \geq 0$ and $j \neq m$.

2. $S_n = A_n$, with $m - j - i \geq 0$ and $j = m$.

3. $S_n = |B_n|$, with $m - j - i < 0$.

1. The result is true for $k = j - i$, since:

$$0 < m - i + \frac{1}{j-i} - 1 \leq m - i$$
$$0 < m - i + \frac{1}{j-i} - 2 \leq m - i - 1$$
$$\cdots$$
$$0 < m - i + \frac{1}{j-i} - (j - i - 1) \leq m - i - (j - i - 2)$$

Therefore:

$$0 < A_{j-i} \leq \frac{1}{j-i}(m-i)\dots((m-i)-(j-i-2)) \leq (m-i)^{j-i-1} \tag{48}$$

As $j > i$ and $(m-i)/(m-j) > 1$ (48) becomes:

$$A_{j-i} \leq \frac{1}{j-i}(m-i)\dots((m-i)-(j-i-2)) \leq \frac{(m-i)^{j-i}}{j-i}\frac{(m-i)^{m-j}}{(m-j)^{m-j}}(j-i)!$$

and therefore:

$$S_{j-i} = A_{j-i} \leq T_{(j,i)}^{j-i}(j-i)! \tag{49}$$

Suppose now, that (47) is true for $k - 1 > j - i$, then, from (38), one gets:

$$A_k \leq A_{k-1}T_{(j,i)}k \leq T_{(j,i)}^k k! \tag{50}$$

as we wish to show. The cases second and third are proven in the same way.

Theorem 4 *Let $H^{(j,i)} : \mathcal{R}^{m-1} \to \mathcal{R}$ be the function:*

$$H^{(j,i)}(x) = \sum_{n=0}^{\infty} \sum_{q_1+\cdots+q_{m-1}=n} d^{(j,i)}(q_1, \cdots, q_{m-1}) x_1^{q_1} \cdots x_{m-1}^{q_{m-1}} \tag{51}$$

where $d^{(j,i)}(q_1, \cdots, q_{m-1})$ are coefficients (20). Then $H^{(j,i)}$ converges uniformly in:

$$D_H^{(j,i)} = \{(x_1, \cdots, x_{m-1}) \in \mathcal{R}^{m-1}, |x_1| + |x_2| + \cdots + |x_{m-1}| < \frac{1}{T_{(j,i)}}\} \tag{52}$$

where $T_{(j,i)}$ is given by (36). Furthermore $H^{(j,i)}$ is a solution of the (j,i)-functional equation of set (11).

Proof. If we consider the absolute value of the series $H^{(i,j)}$, then from (29) and (47) we arrive at:

$$\left|H(x)^{(i,j)}\right| \leq \sum_{n=0}^{\infty}(|x_1|+\cdots+|x_{m-1}|)^n \frac{S_n}{n!} \leq \sum_{n=0}^{\infty}(|x_1|+\cdots+|x_{m-1}|)^n T_{(j,i)}^n$$

From the second inequality uniform convergence is followed. The second part of this Theorem is deduced from Theorem 2.

As a Corollary of Theorem 4, we introduce the following:

Theorem 5 *The Taylor series of the functions, $(f_P^{(j,i)})_v(y)$, $1 \leq v \leq j-i$, introduced in (14), are given by:*

$$\begin{aligned}(f_P^{(j,i)})_v(y) &= k_v^{(j,i)}(y)H^{(j,i)}(X_1^{(j,i)})_v(y),\ldots,(X_{m-1}^{(j,i)})_v(y)) \\ &= (k_v^{(j,i)})_v(y)\sum_{n=0}^{\infty}\sum_{q_1+\cdots+q_{m-1}=n} d^{(j,i)}(q_1,\ldots,q_{m-1})\left((X_1^{(j,i)})_v(y)\right)^{q_1}\cdots\left((X_{q_{m-1}}^{(j,i)})_v(y)\right)^{q_{m-1}}\end{aligned} \tag{53}$$

And they satisfy:
1. Each $(f_P^{(j,i)})_v$, with $1 \leq v \leq j-i$, is uniformly convergent in the region:

$$D_{f_P^{(j,i)}} = \left\{y \in \mathcal{R}; \left|(X_1^{(j,i)})_v(y)\right| + \cdots + \left|(X_{m-1}^{(j,i)})_v(y)\right| < \frac{1}{T_{(j,i)}}\right\} \tag{54}$$

(that does not depend on the chosen subscript v), where the functions $(X_p^{(j,i)})_v$, $1 \leq p \leq m-1$, $1 \leq v \leq j-i$ were defined in (15) or (16), and $T_{(j,i)}$ was introduced in (36).
2. Each $(f_P^{(j,i)})_v$, with $1 \leq v \leq j-i$, is an inverse function of $P(x)$, defined in (12), for all $y \in D_{f_P^{(j,i)}}$.
3. If $0 \in D_{f_P^{(j,i)}}$, then the (j-i) series, given by the expression:

$$\begin{aligned}&(f_P^{(j,i)})_v(0) \\ &= (k_v^{(j,i)})_v(0)\sum_{n=0}^{\infty}\sum_{q_1+\cdots+q_{m-1}=n} d^{(j,i)}(q_1,\ldots,q_{m-1})\left((X_1^{(j,i)})_v(0)\right)^{q_1}\cdots\left((X_{q_{m-1}}^{(j,i)})_v(0)\right)^{q_{m-1}}\end{aligned} \tag{55}$$

converge to $(j-i)$ roots of $P(x)$ respectively, one for each root $k_v^{(j,i)}(0)$.

4 Examples

In this section we give some examples in oder to illustrate the above results.

Example 3 *Consider the polynomial equation $P(x) = 25x^5 + 2x^4 - 8x^3 + 5x^2 + x + 10 = 0$. From (53), with $j = 5$ and $i = 0$, it follows that the five functions, $(f_P^{(5,0)})_v(y)$, $1 \leq v \leq 5$, defined as:*

$$(f_P^{(5,0)})_v(y) = \sqrt[5]{\frac{10-y}{-25}}\sum_{n=0}^{\infty}\sum_{q_1+\cdots+q_4=n} d^{(5,0)}(q_1,\ldots,q_4)\left((X_1^{(5,0)})_v(y)\right)^{q_1}\cdots\left((X_4^{(5,0)})_v(y)\right)^{q_4}$$

where:

$$d^{(5,0)}(q_1,\ldots,q_4) = \frac{q_1+2q_2+3q_3+4q_4-4}{5}\cdots\left(\frac{q_1+2q_2+3q_3+4q_4-4}{5}-(n-2)\right)/(q_1!\ldots q_4!)$$

$$(X_1^{(5,0)})_v(y) = \frac{1}{(-25)}\left(\frac{10-y}{-25}\right)^{-4/5}, \quad (X_2^{(5,0)})_v(y) = \frac{5}{(-25)}\left(\frac{10-y}{-25}\right)^{-3/5},$$
$$(X_3^{(5,0)})_v(y) = \frac{-8}{(-25)}\left(\frac{10-y}{-25}\right)^{-2/5}, \quad (X_4^{(5,0)})_v(y) = \frac{2}{(-25)}\left(\frac{10-y}{-25}\right)^{-1/5}$$

they are the inverse functions of $y = P(x)$ *around its five roots, since* $y = 0$ *belongs to* $D_{f_P}^{(5,0)}$, *that in this example is:*

$$D_{f_P}^{(5,0)} = \left\{ y \in \mathcal{R}; \left| \frac{1}{(-25)}\left(\frac{10-y}{-25}\right)^{-4/5} \right| + \cdots + \left| \frac{2}{(-25)}\left(\frac{10-y}{-25}\right)^{-1/5} \right| < 1 \right\}$$

Therefore, the five roots of $P(x)$ *are given exactly by* $r_v = (f_P^{(5,0)}(0))_v$, $1 \le v \le 5$, *respectively.*

Example 4 *Consider the polynomial equation* $P(x) = 2x^5 + 2x^4 - 40x^3 + x^2 + 5x + 10 = 0$. *Proceeding as in the preceding example, the five roots of* $P(x)$ *are given exactly by* $r_v = (f_P^{(5,3)}(0))_v$, $1 \le v \le 2$ *and* $r_u = (f_P^{(3,0)}(0))_u$, $1 \le u \le 3$ *respectively.*

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